The Flexural Rigidity of Bamboo:
Its application in some rod engineering calculations

Frank Stetzer

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1 The Problem

Rodmakers continue to apply material and mechanical engineering analyses to the design and evaluation of bamboo rods. These applications have, by and large, treated bamboo as it it were a homogeneous material like metal or hardwood, while we know from a century of experience (and a few laboratory analyses) that the properties of our material vary not only from culm to culm, but between different sections of the same culm, between nodal and internodal regions, around the circumference of the culm, and perhaps most significantly, moving from the outer enamel surface toward the inner pith. This is due to the decline in density of the power fibers from enamel to pith that we can see in most culms.

This monograph is concerned with the the decline in the stiffness of bamboo as we move in from the enamel toward the pith. We will investigate this effect of this property on some common engineering analyses performed by rod builders: rod deflection, hollow building, and stress analysis a la Garrison.

What I am doing here is taking what we know and formalizing it. I am not an engineer, and I will not be pushing the envelope of engineering analyses. I am a statistician by profession, so I am used to evaluating data and solving computational problems.

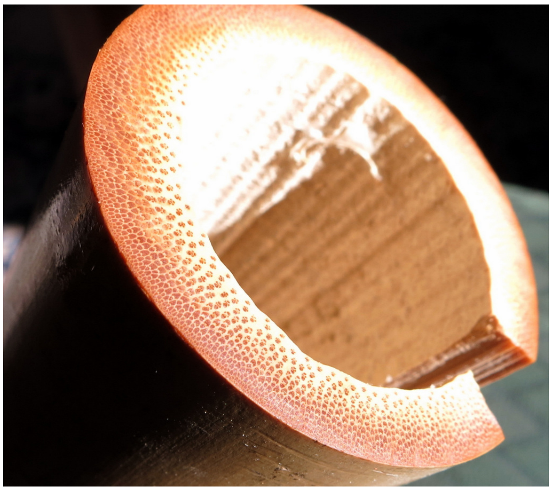

Figure 1: Declining power fibers with depth from enamel

2 Some Engineering Quantities

2.1 Moment of Inertia

An engineering concept commonly used in rod design is the cross section Modulus of Inertia (MOI), usually represented by I. MOI captures the stiffness of a beam (or other structure) due to its cross-sectional shape. It does not depend on the material; the MOI of a hexagonal cross section of a given size will be the same whether the material bamboo, steel or balsa wood. The formula for MOI of a regular beam (a *prismatic beam* in engineering parlance, like a solid hex, penta or square cross section rod) is

$$I = ad^4 \tag{1}$$

where d is the flat-to-flat dimension and a is a constant that depends on the cross sectional shape. For hexagonal cross section the constant a is approximately 0.0601.

Table 1. Constants for Shape in MOI Equations

Shape	d	a	Cross-section Area
Hex	flat-to-flat	0.0601	$0.8660d^2$
Penta	flat-to-apex	0.0427	$0.7206d^2$
Quad	flat-to-flat	0.0866	d^2

The implication of the MOI formula, depending on the rod dimension raised to the fourth power, is that a small increase in rod dimension is magnified to a large increase in stiffness. For instance, a change in dimension from 0.250 to 0.260 (4%) results in a 15% increase in cross sectional stiffness.

The way MOI is used in rod design is typically to design a new taper which is equal in MOI to an existing rod. For example, a quad or penta with the same MOI as a hex, or a hollow rod with the same MOI as a solid rod. The MOI is not a single value for the rod, but a value that can be calculated at any point along its length, depending only on the dimension at that point. In practice we might calculate it at every inch, producing a new taper that is equivalent to the old at each inch point.

2.2 Modulus of Elasticity

The Modulus of Elasticity (MOE) (also called the elastic modulus or Young's Modulus, usually represented by E) is a property of a material that describes its resistance to bending. It depends solely on the material itself, not on its shape. Units for MOE are pounds per square inch (psi).

Robert Milward (2010) and Wolfram Schott (2006) have published laboratory measurements of the MOE of cane from different points in the culm and different sorts of culms. Here I am working from the data in Milward (p 247); Schott's results appear very similar. Milward sampled four culms: "fat regular," "thin regular," " fat Blue Mark," and "thin Blue Mark" [1] From each culm, Milward made from four to eight slices approximately 0.2 inches wide and 0.125 inches thick starting 0.004 inches below the enamel. This sampling procedure was performed in the tip, mid and butt sections of the fat culms and the tip and butt regions of the thin culms, along with the nodal region of one culm for a total of ten sample regions. MOE was determined in a university engineering laboratory.

As can be seen in Figure 2, most of the 10 sample regions exhibits very similar linear declines in MOE as slices progress in from the enamel. A linear regression (the black line in

[1] Blue Mark culms were marked by Charles Demarest and were cut about 3 feet higher from the ground than normal.

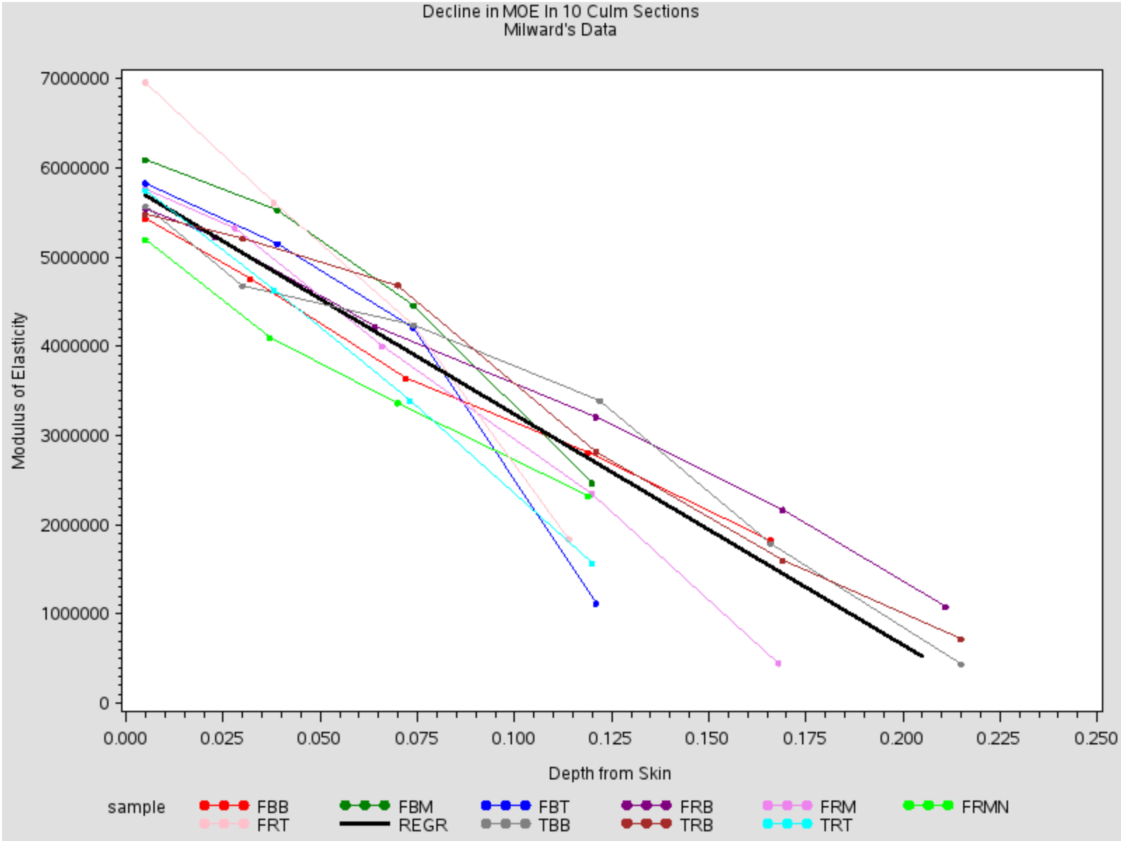

Figure 2: Decline in MOE with depth from skin.

Figure 2) was used to calculate an average relationship between MOE in the strip (E_S) and depth from the enamel (r) in inches:

$$E_S(r) = 5,696,815 - 25,815,764r \tag{2a}$$

In words, the MOE just under the enamel averages 5.7 million psi, and decreases 26,000 psi for each additional 1/1000 of an inch. Applying the formula, at depth of 0.100 inches from the skin, the typical MOE of bamboo is estimated to be $5699815 - 25815764 \times 0.010 = 3115215$. MOE drops to 50% of the theoretical maximum at 0.110 inches from the skin.

Equation 2a gives the decline in MOE with distance from the enamel for a single strip of cane. By a sequence of steps described in the Computation section, it was determined that average or effective MOE for a built-up rod section of any geometry declines at 1/10$^{\text{th}}$ the rate of the single strip. The resulting regression equation for average MOE for a given rod dimension d is

$$E(d) = 5,696,815 - 2,581,576.4d \tag{2b}$$

For example the average MOE of the cane in a rod section of dimension 0.200 is estimated to be $5699815 - 2581576.4 \times 0.200 = 5183495$ which is a 9% decline over the theoretical maximum. This equation is valid only for solid rod construction where the strips are planed as close to the skin as possible.

The apparent precision of these equations is misleading: they are based on a small sample and there is a lot of natural variability in cane that is not represented in this or any practical sample. However they are the best we have at the moment, and I will use these estimated regression equations in the analyses that follow.

The implication of the regular decline in MOE for a finished rod is not surprising. The farther toward the rod butt we go, the more lower MOE cane is in our strips and the less stiff our rod sections will be, compared to what they could be if the power fibers were as dense throughout as they are next to the enamel.

I will use the terms *declining MOE* and *uniform MOE* to refer to these models. Just to be sure the point is clear, declining MOE is reality; uniform MOE is an oversimplification, and, if it is uniformly set at the highest level, misleading.

2.3 Flexural Rigidity

A more complex use of MOI and MOE is in the calculation of rod deflection[2] If we model a short segment of a rod as a cantilever beam, the deflection of the free end under load is given by the formula

$$\delta = \frac{PL^3}{3EI} \tag{3}$$

where δ is the deflection, P is the downward force on the end of the beam (sometimes represented by F), L is the length of the beam, and E and I are the MOE and MOI. In this case we are comparing short segments of rods under the same force, so the numerator is fixed for our purposes. The term EI in the denominator is known as the *flexural rigidity* (B). It combines the stiffness due to the material and the stiffness due to the cross sectional shape.

If bamboo were uniform in MOE, E would be just another constant and the deflection at a point inversely related to just the geometry (a) and rod dimension d (to the fourth power) at that point.

$$B(d) = EI(d) = Ead^4 \tag{4a}$$

But MOE is not uniform, depending on the depth within the culm from which the cane comes. This makes E a function of strip depth d, (as modeled in regression equation 2b) and we have in algebraic terms

$$B(d) = E(d)I(d) = E(d)ad^4 \tag{4b}$$

As the dimension of the rod increases, the MOI and MOE components work in opposite directions. MOI increases very rapidly with the dimension to the fourth power, while the MOE term decreases with dimension at a much lower linear rate. This has led to Milward among others to conclude that the MOE decline can be ignored[3]. In the next section we will quantify this relationship.

3 Application to Rod Taper Design

The flexural rigidity for rod dimensions were computed assuming a uniform MOE of 5.697 Mpsi (the value from regression equation 2b evaluated just under the enamel where power fibers are most dense ($r = 0$)), and also for MOE declining as modeled in equation 2b. The shape constant $a = 0.0601$ for a hexagonal cross section was used, but the results will be proportional for other geometries. The method of computation is described at the end of this paper.

[2]We are considering here the deflection of a short (local) segments of rod (perhaps an inch), where one end is considered fixed (anchored) and the other end free to move under force. The deflection of the entire rod as a whole is a more complex engineering problem and is considered in section 3.4 below.

[3]My results for the decline in flexural rigidity differ significantly from Milward's. This discrepancy will also be discussed at the end.

Table 2. Flexural rigidity for uniform and declining MOE			
	Flexural Rigidity B		
d	E uniform	E declining	% Difference
0.050	2.16	2.09	3.2
0.100	34.2	32.7	4.4
0.150	173	162	6.4
0.200	548	498	9.1
0.250	1337	1186	11.3
0.300	2773	2397	13.6
0.350	5138	4324	15.8

These results do not have an immediate impact on rod taper design (the declining MOE has already been factored into our time-tested rod tapers), but it does show the degree of decline in stiffness with dimension. For a typical trout rod, the butt area is about 12-15% less stiff than it would be with maximum power fibers. This does seem to support the practice of matching the depth of power fibers to the size of rod being built. It also hints at the large gain in stiffness to be achieved by double building. Figure 3 shows the pattern graphically.

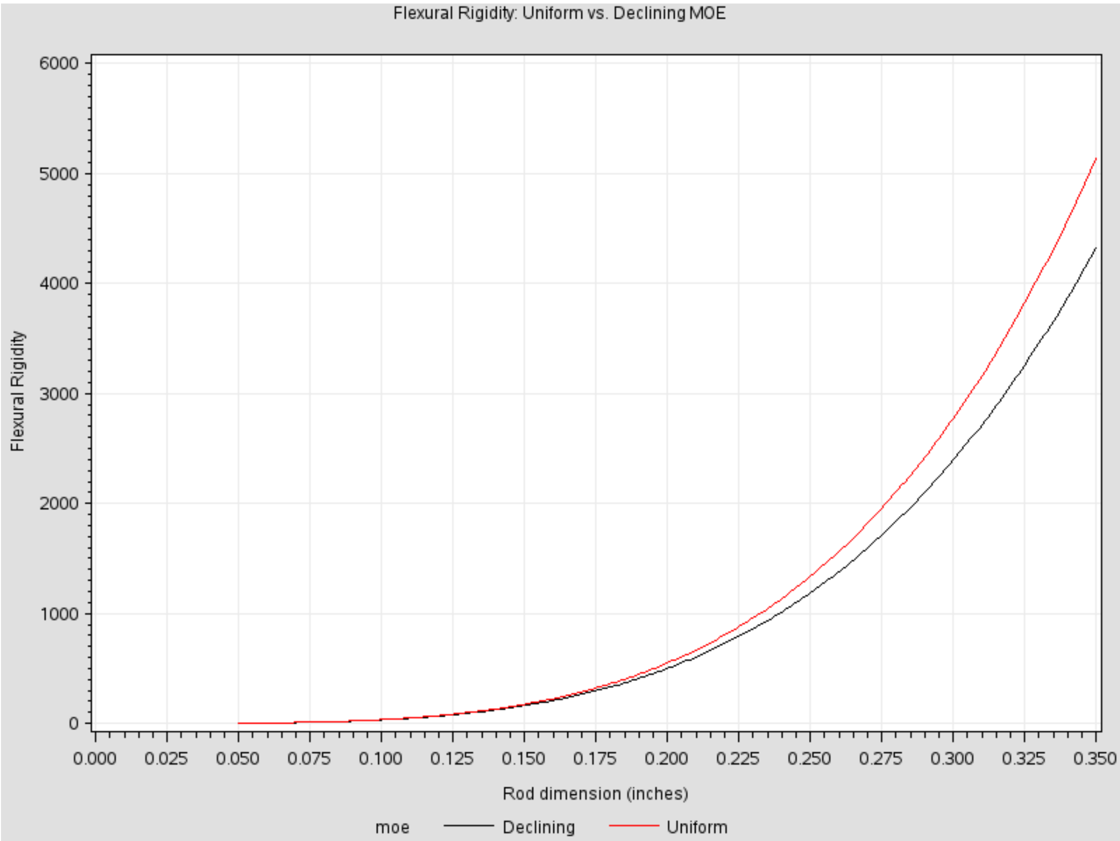

Figure 3: Flexural Rigidity: Uniform and declining MOE compared.

3.1 Altering rod geometries

A potential application of flexural rigidity analysis is in converting tapers from one geometry to another. For each point along the rod (say one inch or five inches), we can determine the dimension in the new geometry that gives the same flexural rigidity (stiffness) as the old. We can do this be either assuming uniform or declining MOE. The Table 3 shows the results for conversion from hex to quad and penta geometries and the simpler conversion by equal cross sectional areas. Cross sectional conversion multipliers come from Table 1: $\sqrt{0.8660} = 0.9306$ for quads and $\sqrt{\frac{0.8660}{0.7206}} = 1.0963$ for pentas.

Table 3. Converting Hex to Quad or Penta Taper

	Equivalent quad d			Equivalent penta d		
Hex d	E uniform	E declining	Equal Area	E uniform	E declining	Equal Area
0.050	0.0461	0.0461	0.0465	0.0545	0.0545	0.0548
0.075	0.0691	0.0691	0.0698	0.0817	0.0818	0.0822
0.100	0.0922	0.0921	0.0931	0.1089	0.1090	0.1096
0.125	0.1152	0.1151	0.1163	0.1361	0.1363	0.1370
0.150	0.1328	0.1380	0.1396	0.1634	0.1637	0.1644
0.175	0.1613	0.1610	0.1629	0.1906	0.1910	0.1919
0.200	0.1843	0.1840	0.1861	0.2178	0.2183	0.2193
0.225	0.2074	0.2069	0.2094	0.2451	0.2457	0.2467
0.250	0.2304	0.2398	0.2326	0.2723	0.2731	0.2741
0.275	0.2534	0.2527	0.2559	0.2995	0.3005	0.3015
0.300	0.2765	0.2756	0.2792	0.3276	0.3280	0.3289
0.325	0.2995	0.2985	0.3024	0.3504	0.3554	0.3563
0.350	0.3226	0.3213	0.3257	0.3812	0.3829	0.3837

For this application, there is basically no difference between using uniform or declining MOE in the flexural rigidity formula. For conversion of hex to quad, the flexural rigidity method proscribes a slightly smaller dimensional rod at the butt end of the rod than the equal area method (e.g. from 0.300 to 0.276 vs. 0.279; about 1/2 line weight at the butt). This seems consistent with the common result that the equal area method results in a slightly faster quad rod than the original hex. For pentas, there is less difference between the two methods.

3.2 Hollowing

Mike McGuire on his website presents the procedure for adjusting rod dimension to compensate for hollowing by scallops and dams. By assuming MOE is uniform, the problem becomes that of finding the hollow rod dimension so that the MOI of the hollow rod equals the MOI of the solid rod. Equation (1) above gives the MOI for a dimensional point d on a solid rod. We can think of a hollow rod as being composed of two parts: an outer rod with dimension d_H and wall thickness t, and an inner " missing rod" of dimension $d - 2t$ which has been removed. The MOI of the solid rod is just the sum of the MOI of the two parts, and the MOI of the hollow rod is the solid rod MOI minus the inner missing rod MOI:

$$I_H = a(d_H^4 - (d_H - 2t)^4) \tag{5}$$

By setting Equation (1) equal to (5), McGuire arrives at a cubic equation which can be solved for d_H in terms of the solid rod dimension d

$$d_H^3 - 3td_H^2 + 4t^2 d_H - 2t^3 - \frac{d^4}{8t} = 0$$

McGuire solves this equation for the five inch stations on Garrison 212 taper. The raising of dimensions to the fourth power ensures that inner missing rod being removed contributes relatively little stiffness compared to the outer hollow rod.

We will extend this analysis to incorporate declining MOE as modeled by equation 2b. Intuitively, if subtracting the inner rod has a relatively small impact on MOI, declining MOE should reduce the stiffness of the inner rod even more, and the impact of hollowing will be less than predicted by McGuire's analysis. The results are shown in Table 4. These results are the same for any rod geometry.

Table 4. Converting Solid to Hollow Rod

	Equivalent hollow d									
	Wall 0.040		Wall 0.060		Wall 0.080		Wall 0.100		Wall 0.120	
Solid d	E uni	E decl	E uni	E decl	E uni	E decl	E uni	E decl	E uni	E decl
0.100	0.1000	0.1000								
0.150	0.1519	0.1516	0.1501	0.1500						
0.175	0.1794	0.1787	0.1754	0.1753	0.1750	0.1750				
0.200	0.2079	0.2066	0.2014	0.2010	0.2001	0.2001				
0.225	0.2374	0.2353	0.2279	0.2272	0.2254	0.2253	0.2250	0.2250		
0.250	0.2679	0.2647	0.2552	0.2539	0.2511	0.2507	0.2501	0.2501	-	-
0.275	0.2994	0.2948	0.2831	0.2811	0.2772	0.2765	0.2754	0.2752	0.2750	0.2750
0.300	0.3318	0.3225	0.3118	0.3087	0.3039	0.3026	0.3010	0.3006	0.3001	0.3001
0.325	0.3651	0.3568	0.3412	0.3368	0.3311	0.3290	0.3269	0.3261	0.3254	0.3252
0.350	0.3991	0.3886	0.3712	0.3653	0.3588	0.3557	0.3532	0.3518	0.3509	0.3504

The increased dimension required to achieve equal stiffness increases with the original dimension and the thinness of the wall, as we would expect. The difference between the assumptions of uniform versus declining MOE are also greatest in the lower left corner of the table. Assuming declining MOE, you do not have to increase rod dimension as much to achieve the same stiffness. This is because the portion of the rod being removed in hollowing is less stiff, so a smaller increase is required to compensate for it.

The relationship between original dimension, wall thickness, and new dimension for the hollow rod is not especially simple; it is displayed in Figure 4 for the case of declining MOE. I have not tried to find a simple equation to capture this pattern.

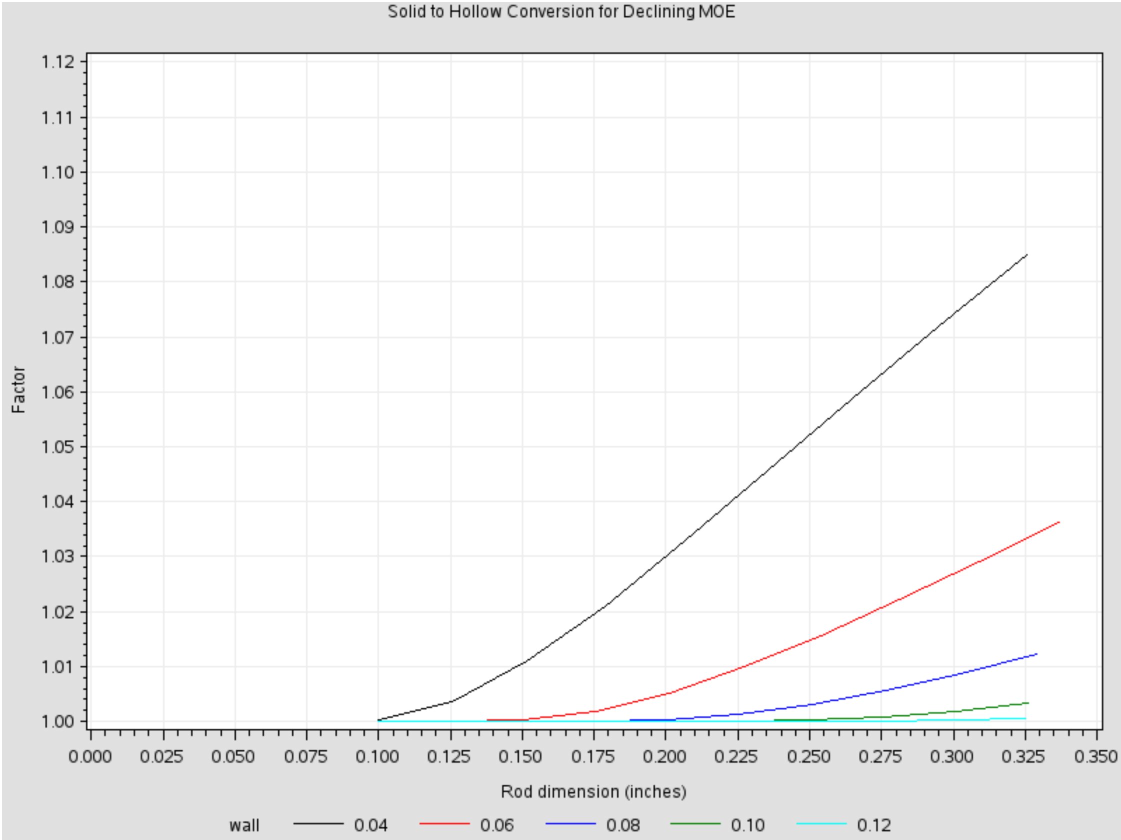

Figure 4: Multiplication Factor for Solid to Hollow Conversion.

3.3 Sanding to Dimension

A common occurrance is for the glued-up rod section to be oversize, even if the planed or milled strips were dead-on the taper measurements. One source of this inflation is the thickness of the glue lines themselves. Ray Gould (2005, p. 33) reports that a typical glue line measures 0.001 inch, and gives the calculations to show that this will result in 0.005 inch increase to a finished hexagonal blank. For a quad and a penta, the increase will be 0.0028. These are for perfectly prepared strips. For hand-planed strips, any irregularity in the angles, which can easily result from the slightest twist or kink or node irregularity, will probably lead to additional dimension inflation.

The formula for MOI informs us that even a slight increase in dimension leads to a much stiffer blank. One solution is to sand or scrape the blank to the desired dimension. Doing this however removes the highest MOE cane, so by the sanding to the original taper dimension, we go too far and end up with lower flexural rigidity than if we could unglue the blank and plane the sides.

We can however use the formulas to compute how much to remove from the blank to achieve the desired flexural rigidity. Equation 2a gives the maximum MOE in a strip from which r outer cane has been removed. If we substitute this for the intercept in 2b, we have an equation for the MOE of a rod section of final dimension d from which r has been removed from all sides:

$$E(d, r) = (5,696,815 - 25,815,764r) - 2,581,576.4d \tag{6}$$

Let d_t be the taper-specified dimension at a point (known), d_m the actual measured dimension at that point (known), and d the dimension we are seeking (unknown). Logically, $d_t < d < d_m$. The amount r to remove from each flat is $r = \frac{d_m - d}{2}$. The flexural rigidity of the taper-specified rod at the point is

$$B = (5,696,815 - 2,581,576.4d_t) \times 0.0601d_t^4 \tag{7}$$

The flexural rigidity of the oversized and sanded rod is

$$B = ((5,696,815 - 25,815,764r) - 2,581,576.4(d_m - 2r)) \times 0.0601(d_m - 2r)^4 \tag{8}$$

We need to set equation 8 equal to 7 and solve for r (or equivalently, solve for d.)

Table 8 below gives the final dimension d after sanding to achieve equal stiffness to a rod with no dimension inflation. This is obviously an large simplification of a complex situation, where the overage is equally distributed around the rod and the adhesive is not adding to the stiffness at all.

Table 5. Sanding to Equal Flexural Rigidity

	Measured Dimension Inflation						
Taper Dim	0.002	0.004	0.006	0.008	0.010	0.012	0.014
0.050	0.0501	0.0501	0.0502	0.0502	0.0503	0.0503	0.0504
0.075	0.0751	0.0752	0.0753	0.0753	0.0754	0.0755	0.0756
0.100	0.1001	0.1002	0.1003	0.1005	0.1006	0.1007	0.1008
0.125	0.1251	0.1253	0.1254	0.1256	0.1257	0.1259	0.1260
0.150	0.1502	0.1503	0.1505	0.1507	0.1509	0.1510	0.1512
0.175	0.1752	0.1754	0.1756	0.1758	0.1760	0.1762	0.1764
0.200	0.2002	0.2005	0.2007	0.2009	0.2011	0.2014	0.2016
0.225	0.2253	0.2255	0.2258	0.2260	0.2263	0.2266	0.2268
0.250	0.2503	0.2506	0.2509	0.2511	0.2514	0.2517	0.2520
0.275	0.2753	0.2756	0.2759	0.2763	0.2766	0.2769	0.2772
0.300	0.3003	0.3007	0.3010	0.3014	0.3017	0.3021	0.3024
0.325	0.3254	0.3257	0.3261	0.3265	0.3269	0.3272	0.3276
0.350	0.3504	0.3508	0.3512	0.3516	0.3520	0.3524	0.3528

As expected, the MOI term with dimension to the fourth power predominates, and we remove most of the extra cane to achieve equal stiffness. For large diameter sections and large inflation, we may want to sand within 0.003 of the taper dimensions, but that is the largest allowance.

3.4 Stress Analysis

The method of stress analysis pioneered by Garrison 40+ years ago remains popular. Similar to MOI, MOE and flexural rigdity, calculation of stress values are made at consecutive points along a rod, typically every inch. Unlike the quantities discussed above, however, stress does not depend on the stiffness due to material or geometry, but simply on the weight bearing on that point along the rod (the moments) from the components of bamboo, line, ferrules, guides and varnish, summed from that point forward to the tip, and usually multiplied by an "impact factor" representing the increased moments that occur during casting. Units of weight for the bamboo are typically ounces per cubic inch, and for stress ounces per square inch and the values are plotted on the familiar stress curve.

Garrison used his stress calculations to design his tapers. His criteria were, first, that a rod casts best when the stress values along a rod are relatively uniform (i.e. stress curve is basically flat, implying all parts of the rod are loading and unloading during the cast), and second, that stress values along the rod should not exceed the safe limit for bamboo which Garrison calculated to be 225,000 oz/in^2.

Since Garrison, critics have brought out many flaws and weaknesses in his method. Milward for one used high speed photography of the cast to calculate actual stress values in the rod and found them to be much higher and distributed very differently than Garrison. He concludes that, as a method to design tapers, "the Garrison stress calculation method is meaningless' (p. 128).

The problem with Milward's criticism is that no one today actually uses stress numbers in this way, designing rods from scratch like Garrison. In practice builders use stress numbers and stress curves to compare and modify rods. By accounting for the weight of the bamboo, line and components, stress curves are believed to capture the "feel" of the rod being cast better than a graph of dimensions alone. Moreover, they provide a systematic way to modify rod tapers to account for changes in line weight, number, type and placement of ferrules, and modest changes in rod length while retaining the same casting feel. The debate will continue.

Since MOE does not enter into stress calculations in any way, whether MOE is considered uniform or declining is not directly relevant. However, MOE is dependent on the density of power fibers in the strip, which affects the weight and enters into stress calculations in that way. Schott (p. 18) presents weights for glued-up hexagonal rod sections of different diameters:

Table 6. Specific Weight of Rod Sections	
Dimension	Specific Weight (oz/in^3)
0.104	0.681
0.160	0.670
0.239	0.651
0.316	0.637

Admittedly a very small sample (based on a single "high-quality culm"), the numbers compare closely with Garrison's measured weight of 0.668 oz/in^3. If we convert these to a linear equation by regression we arrive at

$$W(d) = 0.703 - 0.211d \tag{8}$$

where d is rod dimension. The relative rate of decline in weight is less than the corresponding decline in MOE from equation 2b. At the 0.150 inch dimension point, MOE in a rod section has declined 6.8% from its theoretical maximum, while weight has declined 4.5%.

If we use this equation to replace the constant 0.668oz/in^3 in the stress calculations, we have the following bamboo moments for a Garrison 212 taper:

Table 7. Bamboo Moments for Garrison 212				
Station	Dimension	Weight 0.668 oz/in^3	Weight from Eqn 6	% Change
5	0.083	0.158	0.163	3.1
10	0.104	0.738	0.757	2.6
15	0.122	1.942	1.986	2.3
20	0.136	4.001	4.081	2.0
25	0.149	7.131	7.255	1.7
30	0.162	11.549	11.721	1.5
35	0.175	17.488	17.706	1.2
40	0.187	25.199	25.453	1.0
45	0.200	34.941	35.212	0.8
50	0.212	46.998	47.253	0.5
55	0.227	61.672	61.864	0.3
60	0.239	79.320	79.382	0.1
70	0.266	124.959	124.468	-0.4
75	0.280	153.737	152.764	-0.6
80	0.295	187.063	185.425	-0.9

Bamboo moments account for typically 1/3 of the total moments going into the stress equation, so the small effect of declining weight becomes minuscule in the stress values and curve.

3.5 Deflection

Some rod makers use calculated deflection of the entire rod as a tool for evaluating and comparing rod tapers. Starting with the formula for deflection of a end-loaded cantilever beam (equation 3 from section 2.3 above):

$$\delta = \frac{PL^3}{3EI} \tag{3}$$

deflection is calculated for small sections of the rod (say, $L =$ one inch), then the deflection is accumulated for the entire rod. The flexural rigidity term $B = EI$ enters the equation in the denominator. Under the assumption of uniform MOE, E is constant and I depends on the dimension of the rod, increasing with the fourth power of dimension d (equation 1). Under decreasing MOE, E decreases according to regression equation 2b, and so the result is that deflection will be greater, and will increase proportional to the third power of dimension. Basically, the difference in the denominator term EI under the two assumptions can be seen in Table 2 and Figure 3.

There is some ambiguity in how to implement equation 3 in calculating entire rod rod deflection. Besides the loading P at the end of the beam, there are other weights involved which contribute to the deflection, basically all the weights incorporated in the traditional stress calculations: bamboo, ferrules, line in guides, and varnish and guides. Under the assumption of declining MOE, we may also want to account for declining bamboo weight with dimension as we did in Section 3.3.

A more complex problem concerns how to accumulate the deflection terms for each short section. The deflection of each segment affects the deflection of other segments. The more segment i deflects, the shorter the rod becomes, reducing the leverage (moment) effects (weight times length) on segments from i toward the butt. And the more segment i deflects, the angle at which deflection starts in segments from i toward the tip increases, which shortens the horizontal length of those segments and again affects the leverage. Correct calculation of deflection becomes an iterative process, operating in both directions along the rod.

On top of this is the fact that equation 3 is really only appropriate for small deflections of a cantilever beam, where the projected or effective length of the beam does not change.

Appropriate formulas (in the form of differential equations) are not easy to solve, although Beléndez et al. (2002) and Chen (2010) provide formulations that can be addressed by numerical integration. These papers can be found with an internet search.

Equation 3 was used to calculate the deflection of a Garrison 212 taper, incorporating a modest weight loading at the end of the rod and the weight of the bamboo but not the other components. Under the assumption of declining MOE and declining bamboo weight with dimension, the deflection increases by about 10% at the tip, not a great difference but one which is easy to incorporate in the otherwise complex calculations.

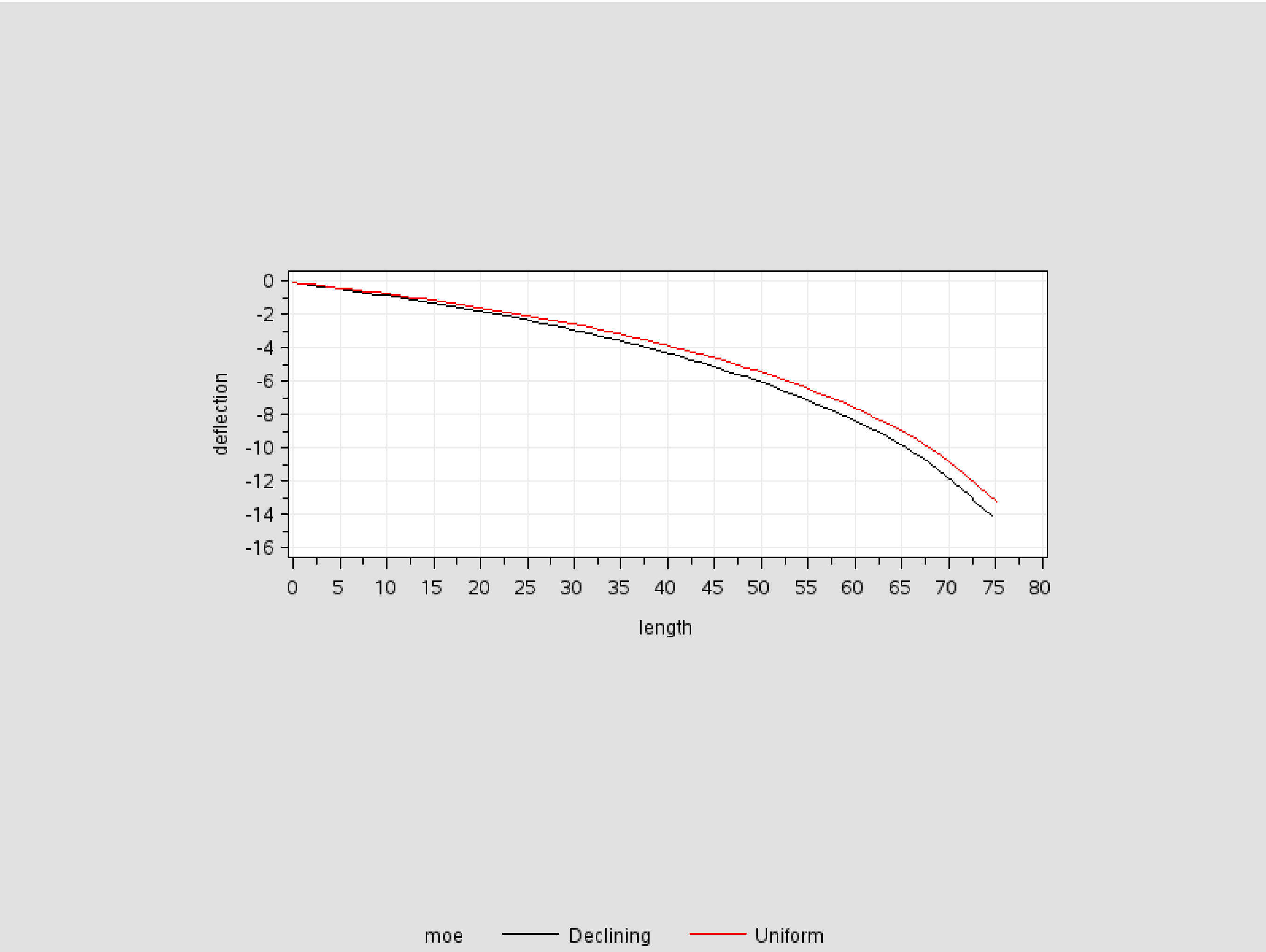

Figure 5: Deflection of Garrison 212 taper under uniform and declining MOE.

4 Additional Sources of Variability

So far we have dealt with only one source of variation in MOE: its fairly regular, predictable decline from the outer skin of a strip toward the pith. Based on Milward's data we have modeled this in equations 2a and 2b.

There are at least three other sources of MOE variability within the cane itself. First, Milward measures and discusses in detail the change in MOE from butt to tip within the culm:

Table 7: Decline in MOE from Culm Tip to Butt (Milward)		
Area	Avg max MOE mpsi	Spread
Tip	6.06	± 8.8%
Mid	5.9	± 2.8%
Butt	5.5	± 1.3%
Avergage	5.78	+14% to -6%

Since we usually build rod tips from culm tips and butts further down, this variation is probably factored into our established tapers. But there is considerably variability especially within the tips; see Milward for details of how this varies among the culms he tested.

Second, there is variability around the circumference of the culm. Schott has studied this in some detail, but based on a single level cut from a single culm. He found a systematic pattern to the distribution of MOE around the culm, with the weakest strip opposite the stiffest, and measuring only 75% of the maximum MOE. If this result generalizes to most culms, it would potentially have a large impact on rod design. Schott develops the significance of this result for the selection and matching of strips in the rod. He does not explore whether the stiffness of the strip can be assessed visually.

Finally, and perhaps most simply, there is variability between culms that we all take note of when we evaluate the depth of power fibers and select a culm for a rod. We have no hard data on how closely visual our judgment of power fibers (usually just at the butt and tip of a whole culm) correlates with the actual MOE. Schott would have builders set up their own apparatus to measure the actual MOE of their cane, but few will probably go that far. If we could easily determine the MOE of our culm or strips in advance, the results above would let us tweak a taper to achieve predictable results.

What I am going to do here, just for the sake of discussion, is to look at the variability of MOE in the three of the four culms analyzed by Milward (those with at least two sample regions) and described in section 2.2 above. The resulting flexural rigidity regression equations for a glued-up section are

$$\begin{aligned} E(d) &= 5,698,815 - 25,816,478d && \text{average; equation 2a} \\ E(d) &= 5,945,319 - 27,442,590d && \text{fat regular culm} \\ E(d) &= 5,892,923 - 28,880,898d && \text{fat Blue Mark culm} \\ E(d) &= 5,563,258 - 24,839,923d && \text{thin regular culm} \end{aligned}$$

It is hard to visualize what these are telling us about a rod potentially built from each, but the table below gives the rod dimensions required from each culm for equal flexural rigidity to the “average” culm. Milward describes all the culms he evaluates as “good,” how this compares to the judgment of other builders is unknown. Most hobby or amateur builders do not have a large stock of culms on hand, nor are we inclined to discard a significant number of culms due to lack of power fibers. Chances are, the culms many of us actually use have much more variation than the three Milward analyzed.

Table 8. Dimensions for Equal Flexural Rigidity			
Average d	Fat Regular Culm	Fat Blue Mark Culm	Thin Regular Culm
0.050	0.0500	0.0500	0.0503
0.075	0.0742	0.0744	0.0754
0.100	0.0989	0.0992	0.1006
0.125	0.1237	0.1241	0.1257
0.150	0.1484	0.1489	0.1509
0.175	0.1732	0.1738	0.1760
0.200	0.1979	0.1987	0.2011
0.225	0.2227	0.2236	0.2263
0.250	0.2474	0.2485	0.2514
0.275	0.2722	0.2734	0.2765
0.300	0.2969	0.2984	0.3017
0.325	0.3217	0.3233	0.3268
0.350	0.3464	0.3483	0.3519

The difference between hypothetical rods built from the stiffest fat regular culm and the weakest thin regular culm is about 0.003 in the lower half of the rod, or about half a line weight. For the wider range of culms we end up using, it is easy to imagine a full line weight difference. The important point is that, if we have a good way to judge the stiffness of our strips, either by direct measurement or evaluation, we can adjust a taper to meet the goal of predictable stiffness.

5 Limitations

In addition to the obvious limitation of scant data, there are several others which must be mentioned.

First, the equations for MOE developed in section 2.2 assume we are plane the strip as close to the enamel as possible. If we scrape off the enamel to flatten the strip, especially a wide strip, or to address a dip near a node etc., we are removing the some of the stiffest part of the strip. Equation 2a implies that that removing 0.010 from the enamel side reduces the outermost stiffness of the strip by about 5%.

Second, the equations developed here assume a constant linear decline in MOE with depth from the outer skin. We have all seen culms where the power fibers appear equally dense for a distance beneath the skin, then rapidly drop off toward the pith. It is not certain that MOE will follow the same pattern, and most of Milward's samples shown in Figure 2 do not show it (except perhaps the fat Blue Mark tip (FBT)), but it may be a source of additional variability.

Third, Milward's MOE numbers come from raw cane. Schott looks at increases in MOE due to heat treating. A portion of the change is due to loss of water and shrinkage, which is partly reversed over time, but he estimates a permanent increase in MOE of about 5%.

Fourth, the stiffness of the glued-up rod sections depends to some extent on the adhesive used. Milward uses urea formaldehyde glue in his tests and Schott used resorcinol. Both remark that these are considered stiff compared to some other common glues.

6 Computational Methods

This section is for those who are interested in the computational details, or who want to check my work.

Computation of flexural rigidity of the hexagonal rod section takes advantage of the fact that, for a beam with a regular (prismatic cross section (like a hex, quad, or penta rod, with equal dimension in all directions) we can consider the rod to be made up of many concentric layers of decreasing MOE. In engineering parlance, this is a *composite prismatic beam.* Figure 6 shows a rod modeled as eight layers[4].

To illustrate the computations, we will use a simple rod comprised of three layers as in Figure 7. The outer layer A is a hollow hexagonal tube with outside dimension 0.300 inch and wall thickness 0.050. Tube B has an outside dimension 0.200 inch and wall thickness 0.050. Section C is a solid hexagonal shape with outside dimension 0.100 inch. Each layer has a MOI I dependent on its dimension

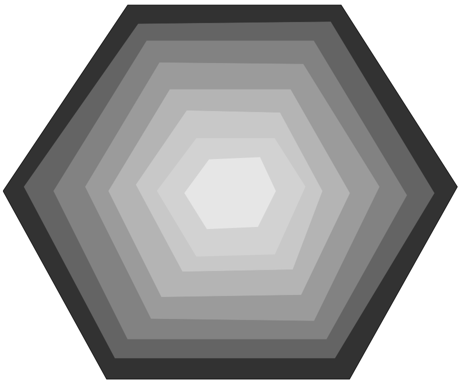

Figure 6: Decling MOE with depth from enamel (a composite beam)

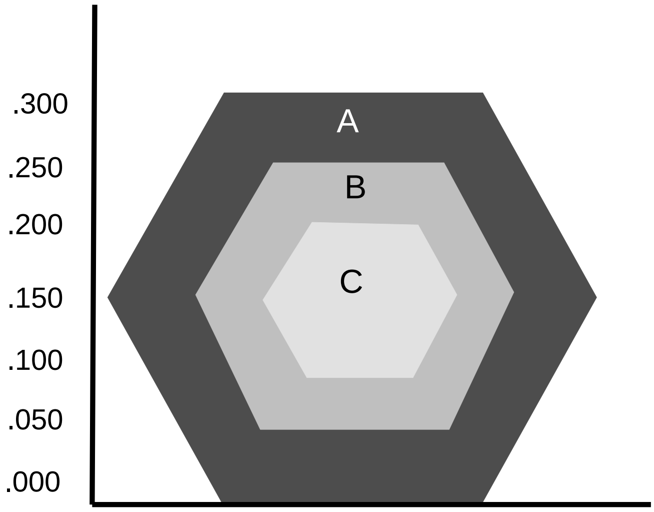

Figure 7: Declining MOE approximated by three layers

If bamboo were uniform in MOE, flexural rigidity is given simply by equation 4a. Using E=5696815 psi, we have

$$B = 5696815 \times 0.0601 \times 0.3^4 = 2773$$

The same R can be determined by calculating B for each layer A, B and C and summing the values:

$$\begin{aligned} B = B_A + B_B + B_C &= EaI_A + EaI_B + EaI_C \\ &= 5696815 \times 0.0601 \times ((0.3^4 - 0.2^4) \\ &\quad +5696815 \times 0.0601 \times (0.2^4 - 0.1^4) \\ &\quad +5696815 \times 0.0601 \times 0.1^4 \\ &= 2225 + 514 + 34 \\ &= 2773 \end{aligned}$$

Now we will let the MOE decline as described in equation 1. For simplicity we will start by assuming, for each layer, the MOE is determined by the depth to the outermost fibers.

[4]No doubt there is an elegant solution to finding the flexural rigidity of a hexagonal composite beam with MOE declining linearly from surface to center. I'm happy to leave this problem to those with better calculus skills.

$$
\begin{aligned}
E_A &= 5696815 \\
E_B &= 5696815 - 25816 \times 50 = 4406015 \\
E_C &= 5696815 - 25816 \times 100 = 3115215
\end{aligned}
$$

Combining the declining MOE of the bamboo and the declining MOI of the hexagonal tubes, we have

$$
\begin{aligned}
B = B_A + B_B + B_C &= E_A a I_A + E_B a I_B + E_C a I_C \\
&= 5696815 \times 0.0601 \times ((0.3^4 - 0.2^4) \\
&\quad +4406015 \times 0.0601 \times (0.2^4 - 0.1^4) \\
&\quad +3115215 \times 0.0601 \times 0.1^4 \\
&= 2225 + 397 + 19 \\
&= 2641
\end{aligned}
$$

We see that the flexural rigidity of the realistic 0.300 inch rod section, based on declining MOE, is only about 4.8 percent lower than a section with maximum MOE. One problem with this calculation is illustrated in figure 8. By estimating the MOE of each layer as the MOE of the outermost part of the layer, we overestimate the MOE of the entire layer. The areas above the regression line and below the three steps represent over-estimation of MOE.

Once we have the flexural rigidity B, we can use equation 4a to compute what I am calling the "effective MOE", E^*, the uniform MOE which results in the same B:

$$
E^* = \frac{2641}{0.0601 \times 0.3^4} = 5425115
$$

by these calculations, the declining MOE results in only a 4.8% decline between maximum and effective MOE.

The problem with this calculation is illustrated in figure 8. By estimating the MOE of each layer as the MOE of the outermost part of the layer, we overestimate the MOE of the entire layer. The areas above the regression line and below the three steps represent over-estimation of MOE.

No doubt there is an elegant engineering solution to this, involving lots of calculus, but it is computationally easy to increase the number of layers of hollow hexagonal tubes used to approximate the continuous decline in MOE with depth. We can just as easily do 10, 100 or 1000 layers as three. That's what computers are good at. Table 9 shows the flexural rigidity as calculated with increasing numbers of layers. (Calculations are done to more decimal places than shown.)

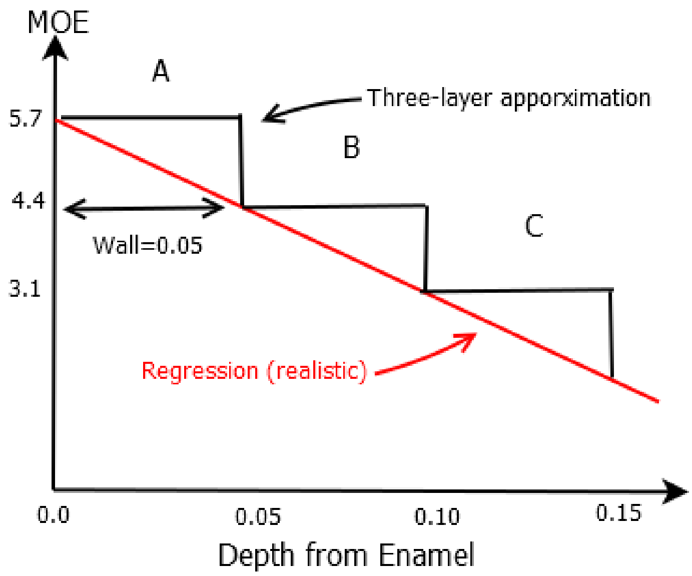

Figure 8: Declining MOE in three-layer model

Table 9. Flexural Rigidity computed with increasing number of layers			
	Rod Dimension		
	0.100	0.200	0.300
$B_{uniform}$	34.2	548	2773
$E_{uniform}$	5696815	5696815	5696815
Wall=0.05			
Layers	1	2	3
Effective E^*	5696815	5616140	5425906
B	34.2	540	2641
Decline from uniform	0%	1.4%	4.8%
Wall=0.01			
Layers	5	10	15
Effective E^*	5550593	5300978	5045681
B	33.4	510	2456
Decline from uniform	2.6%	6.9%	11.4%
Wall=0.001			
Layers	50	100	150
Effective E^*	5451391	5193317	4935186
B	32.8	499	2402
Decline from uniform	4.3%	8.8%	13.4%
Wall=0.0001			
Layers	500	1000	1500
Effective E^*	5431944	5181765	4923625
B	32.7	498	2397
Decline from uniform	4.5%	9.0%	13.6%

We see that the correct computation of rigidity under conditions of declining MOE depends on the granularity of our analysis. Looking at the 0.300 inch point, assuming uniform (maximum) MOE gives a rigidity of 2773, using 3 layers of thickness 0.05in gives 2641, 15 layers of 0.01 thickness gives 2456, 150 layers of 0.001 thickness gives 2402, and finally 1500 layers of thickness 0.0001 gives a rigidity of 2397, a decline of 13.6% from the uniform value. Values presented in Table 2 and Figure 3 are based on a wall thickness of 0.0001 inch.

The exact linear relationship between rod dimension and effective MOE as modeled in regression equation 2b came as a surprise to me; I was expecting a more complex nonlinear relationship. I have not attempted to derive 2b from 2a but I'm sure it could be done.

7 Comparison with Milward's analysis

One reason I have presented the calculation steps in detail because my conclusions differ from Milward's, and although I have an idea where the problem lies it is possible I am missing something important.

Milward finds a reduction in stiffness due to declining MOE of 5.0% for a dimension of 0.300 inches, 4.3% for 0.200, and 2.7% for 0.100. He concludes "The reduction of the MOE with depth below the bamboo skin has a comparatively minor effect on the total stiffness and very little effect on the distribution of build-up stiffness through the section." (2010, p 102) My results are considerably larger: 13.6%, 9.0%, and 4.5% (from bottom of Table 6). These effects seem large enough to me that they should not be ignored in calculating and applying rod stiffness.

I will go through Milward's calculations step-by-step. He begins with equation (3) for deflection of a hexagonal cantilever beam of dimension d under the assumption of uniform MOE, and rearranges it to find Force (P) for a fixed deflection δ, MOE E, and beam length L:

$$P = \frac{\delta 3EI}{L^3} \tag{9}$$

where P = load in pounds, $\delta = 0.02$ inches, $E = 5700000$ psi, $I = 0.60d^4$, $L = 1$ inch. Combining the constants we have

$$P = 20520d^4 \tag{9}$$

The product EI is flexural rigidity B. Ignoring the small differences in maximum MOE (5,700,000 vs. 5,696,815) and geometry constant a (0.060 vs. 0.0601), the relationship between Milward's force P and flexural rigidity B is just a multiplier:

$$16.667P = B$$

P and B are equivalent measures of stiffness.

Milward uses the calculations in his Table 5.5 (2010, p. 250) to make his case that the outer fibers of the rod do the vast majority work. He uses the technique I employ here by treating the rod as if it were composed of layers for which the quantity in question (P) can be calculated and summed. The left side of the table is straightforward: it shows the force required to produce the desired deflection for layers of different thickness. Here is the top of the table:

Milward's Table 5.5: Left hand side					
At 0.300 inch diam station				**Assume uniform E**	
Wall				P (lbs)	Strength
0.010	$P = P_{300} - P_{280} =$	166.21 -	126.126 =	40.08	
0.020	$P = P_{300} - P_{260} =$	166.21 -	93.770 =	72.43	
0.030	$P = P_{300} - P_{240} =$	166.21 -	68.079 =	98.13	
0.040	$P = P_{300} - P_{220} =$	166.21 -	48.068 =	118.14	71%
0.050	$P = P_{300} - P_{200} =$	166.21 -	32.838 =	133.38	80%
0.060	$P = P_{300} - P_{180} =$	166.21 -	21.539 =	144.67	87%
0.070	$P = P_{300} - P_{160} =$	166.21 -	13.447 =	152.76	92%
		etc.			

It is the right side of the table incorporating declining MOE that I find problematic. Adding the columns not shown:

Milward's Table 5.5: Right hand side					
At 0.300 inch diam station				**Assume E diminishes at 6.6% / 0.010″**	
Wall				P (lbs)	Strength
0.010	$P = P_{300} - P_{280} =$	157.87 -	117.801 =	40.08	
0.020	$P = P_{300} - P_{260} =$	157.87 -	87.581 =	70.29	
0.030	$P = P_{300} - P_{240} =$	157.87 -	63.586 =	94.29	
0.040	$P = P_{300} - P_{220} =$	157.87 -	44.896 =	112.98	71.6%
0.050	$P = P_{300} - P_{200} =$	157.87 -	30.671 =	127.21	80.6%
0.060	$P = P_{300} - P_{180} =$	157.87 -	20.117 =	137.75	87%
0.070	$P = P_{300} - P_{160} =$	157.87 -	12.559 =	145.31	92%
		etc.			

The first question is, how was the force value of 157.87 lbs arrived at? Milward does not show his calculations, but here is one possibility. In the right hand side table, the force of 166.21 for the rod dimension 0.300 and the force of 126.126 for the 0.280 rod are computed directly from equation 9, giving a difference of 40.08 lbs for the outer layer of wall thickness 0.010. In the left hand table, under the assumption of declining MOE, the outer 0.010 layer will still be (about) equal strength of 40.08. The inner 0.280 dimension rod will have less stiffness, reduced 6.6% from the uniform value of 126.126, or 117.79 lbs. This makes the stiffness of the entire 0.300 rod 40.08+117.69=157.87.

The problem with this approach is that the stiffness of the inner 0.280 dimension rod itself is declining from outer to inner layers. The value of 117.79 is an overestimate of its stiffness, just like the three-layer model in Figure 8 overestimated MOE. My calculations result in a value of 104.55 lbs.

A second problem with Milward's calculations arise in subsequent rows. Assuming Milward's E decline of 6.6% per 0.010 inch(= 376,200, higher than my rate of about 4.5% per 0.010 or 258,160 from equation 2a; I cannot find where Milward calculates the 6.6%), the E value should decline more the farther from the enamel we go. This does not affect the calculation of the overall stiffness drop due to declining MOE, but will change the percentage values in the Strength column.

Here is the right half of Milward's table as I calculate it:

Milward's Table 5.5: Left hand side recalculated					
At 0.300 inch diam station					**Assume E diminishes at 258,160 / 0.010″**
Wall				P (lbs)	Strength
0.010	$P = P_{300} - P_{280} =$	143.81 -	104.552 =	39.26	27%
0.020	$P = P_{300} - P_{260} =$	143.81 -	74.328 =	69.48	48%
0.030	$P = P_{300} - P_{240} =$	143.81 -	51.493 =	92.32	64%
0.040	$P = P_{300} - P_{220} =$	143.81 -	34.613 =	109.20	76%
0.050	$P = P_{300} - P_{200} =$	143.81 -	22.449 =	121.36	80%
0.060	$P = P_{300} - P_{180} =$	143.81 -	13.947 =	137.75	90%
0.070	$P = P_{300} - P_{160} =$	143.81 -	8.219 =	13.59	94%
		etc.			**13.6% MOE loss**
At 0.200 inch diam station					
0.010	$F = P_{200} - P_{180} =$	29.897 -	18.834 =	11.054	37%
0.020	$P = P_{200} - P_{160} =$	29.897 -	11.270 =	18.627	62%
0.030	$P = P_{200} - P_{140} =$	29.897 -	6.320 =	22.557	79%
0.040	$P = P_{200} - P_{120} =$	29.897 -	3.257 =	26.640	89%
0.050	$P = P_{200} - P_{100} =$	29.897 -	1.496 =	28.401	95%
		etc.			**9.0% MOE loss**
At 0.100 inch diam station					
0.010	$P = P_{100} - P_{080} =$	1.962 -	0.773	1.189	61%
0.020	$P = P_{100} - P_{060} =$	1.962 -	0.235	1.727	88%
0.030	$P = P_{100} - P_{040} =$	1.962 -	0.044	1.918	98%
		etc.			**4.5% MOE loss**

8 References and Resources

R. E. Milward. *Bamboo: Fact, Fiction and Flyrods - II*. 2010.
Both the 2001 and 2010 versions contain the information on bamboo MOE used here. The 2010 version is available as a download for purchase on Milward's website http://www.bobmilwardbamboo.com.

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Tarsicio Beléndez, Cristian Neipp and Augusto Beléndez. Large and small deflections of a cantilever beam. *European Journal of Physics 23(3)*; May 2002, pp. 371-9.

Li Chen. An integral approach for large deflection cantilever beams. *International Journal of Non-Linear Mechanics 45*; 2010, pp 301-5.
See especially section 3.4 for analysis of a tapered cantilever beam.