Rodmaker's Geometry Reference

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November 1, 2019; Last update: November 1, 2020

1 Purpose

This paper is a quick reference guide to some common geometric and trigonometric calculations used in the design and building of split cane rods. Nothing here is new or advanced in any way, but it may save you doing the derivations and calculations from scratch. Four, five and six sided rods are considered.

I have checked the derivations and calculations by doing each twice, but it is possible that I made the same mistake twice. If you find a problem PLEASE let me know so it can be corrected or improved.

2 Basics

Rods are constructed of four, five or six strips that are equilateral or isosceles triangles in cross section. For geometric calculations I find it most convenient to consider each cross section to be composed of two identical right triangles with known angles. From a right triangle with known angles and known length of one side, it is posible to find the other lengths by simple trigonometry. This will be my method. There might be more than one formulation and solution of some of the problems. The right triangles and their angles are shown in Figure 1.

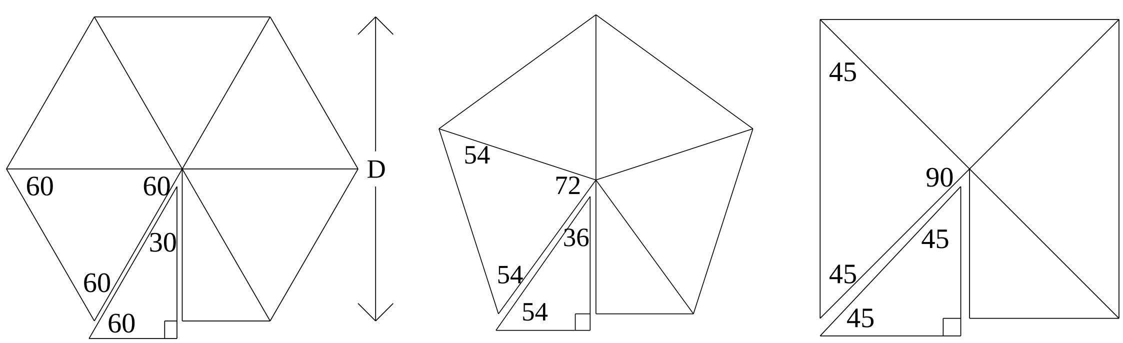


Figure 1: Rod geometries and associated right triangles

The basic taper measurement is the dimension D at a station. For the hexagonal and quadrate constructions, the dimension is from flat to flat; for pentagonal it is from flat to apex. From D and

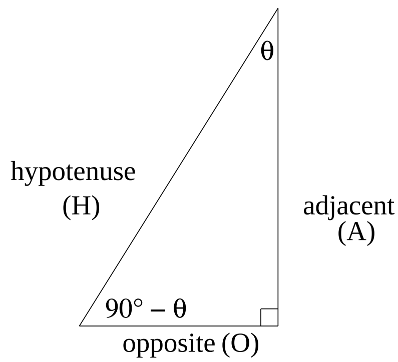


Table 1. Basic trig functions

$\sin\theta$	$=$	$\frac{opposite}{hypotenuse}$	$=$	$\frac{O}{H}$
$\cos\theta$	$=$	$\frac{adjacent}{hypotenuse}$	$=$	$\frac{A}{H}$
$\tan\theta$	$=$	$\frac{opposite}{adjacent}$	$=$	$\frac{O}{A}$

the right triangle, most of the basic quantities can be derived.

Figure 2: Right triangle and associated trig functions

Table 2. Trig function values for useful angles

Degrees	Radians	sin	1/sin	cos	1/cos	tan	1/tan
18	$\frac{\pi}{10}$ (0.3142)	0.3090	3.2361	0.9511	1.0515	0.3249	3.0777
30	$\frac{\pi}{6}$ (0.5236)	0.5000	2.0000	0.8660	1.1547	0.5774	1.7321
36	$\frac{\pi}{5}$ (0.6283)	0.5878	1.7013	0.8090	1.2361	0.7265	1.3764
45	$\frac{\pi}{4}$ (0.7854)	0.7071	1.4142	0.7071	1.4142	1.0000	1.0000
54	$\frac{3\pi}{10}$ (0.9425)	0.8090	1.2361	0.5878	1.7013	1.3764	0.7265
60	$\frac{\pi}{3}$ (1.0072	0.8660	1.1547	0.5000	2.0000	1.7321	0.5774
72	$\frac{2\pi}{5}$ (1.2556)	0.9511	1.0575	0.3090	3.2361	3.0777	0.3249
90	$\frac{\pi}{2}$ (1.5708)	1.0000	1.0000	0.0000	∞	∞	0.0000

3 Strip and rod dimensions

The information in the section above lets us derive basic results about the dimensions of the rod and component strips in terms of the rod dimension D. When describing strips, the term *enamel* refers to the middle of the outside surface, *apex* refers to the point where the planed surfaces meet at the center of the rod (opposite the enamel), and *corner* refers to the point where the enamel meets the planed side. A, O and H refer to the length of the adjacent, opposite and hypotenuse sides of the right triangle in Figure 2. Table 3 is on the following page.

Table 3. Strip and rod measurements

Geometry	Measure	Formula			Compute	
Hexagonal	Right triangle sides	A	$=$	$\frac{D}{2}$	$0.5D$	(1)
		H	$=$	$\frac{D}{2\cos 30}$	$0.5774D$	(2)
		O	$=$	$\frac{\tan 30}{2}D$	$0.2887D$	(3)
	Strip enamel $\rightarrow$ apex	A	$=$	$\frac{D}{2}$	$0.5D$	(4)
	Strip corner $\rightarrow$ apex	H	$=$	$\frac{D}{2\cos 30}$	$0.5774D$	(5)
	Strip corner $\rightarrow$ corner	$2O$	$=$	$\frac{D}{2\cos 30}$	$0.5774D$	(6)
	Rod corner $\rightarrow$ corner	$2H$	$=$	$\frac{D}{\cos 30}$	$1.1547D$	(7)
Pentagonal	Right triangle sides	A	$=$	$(1 - \frac{1}{1+\cos 36})D$	$0.4472D$	(8)
	$(D = A + H)$	H	$=$	$\frac{D}{1+\cos 36}$	$0.5528D$	(9)
		O	$=$	$\frac{\sin 36}{1+\cos 36}D$	$0.3249D$	(10)
	Strip enamel $\rightarrow$ apex	A	$=$	$(1 - \frac{1}{1+\cos 36})D$	$0.4472D$	(11)
	Strip corner $\rightarrow$ apex	H	$=$	$\frac{D}{1+\cos 36}$	$0.5528D$	(12)
	Strip corner $\rightarrow$ corner	$2O$	$=$	$\frac{2\sin 36}{1+\cos 36}D$	$0.6498D$	(13)
	Rod corner $\rightarrow$ corner			$\frac{4\sin 36\cos 36}{1+\cos 36}D$	$1.0515D$	(14)
Quadrate	Right triangle sides	A	$=$	$\frac{D}{2}$	$0.5D$	(15)
		H	$=$	$\frac{D}{2\cos 45}$	$0.7071D$	(16)
		O	$=$	$\frac{D}{2}$	$0.5D$	(17)
	Strip enamel $\rightarrow$ apex	A	$=$	$\frac{D}{2}$	$0.5D$	(18)
	Strip corner $\rightarrow$ apex	H	$=$	$\frac{D}{2\cos 45}$	$0.7071D$	(19)
	Strip corner $\rightarrow$ corner	$2O$	$=$	D	D	(20)
	Rod corner $\rightarrow$ corner	$2H$	$=$	$\frac{D}{\cos 45}$	$1.4142D$	(21)

4 Strip angles from dimensions

The situation can arise during planing when things go amiss and we need to determine the actual angles at a station. Mike McGuire shared the formulas to solve for the angles from the dimensions of the strip. In Figure 3 below, A, B and C are the unknown angles of the verticies; a, b and c are the sides opposite those verticies, and h_a, h_b and h_c are the triangle altitudes, the distance measured with a micrometer or caliper from the side to the opposite vertex. Using formulas for triangle area and the Law of Cosines, Table 4 gives the resulting formulas involving the $arccos$ function; you will need a scientific calculator or computer spreadsheet to complete the calcuations.

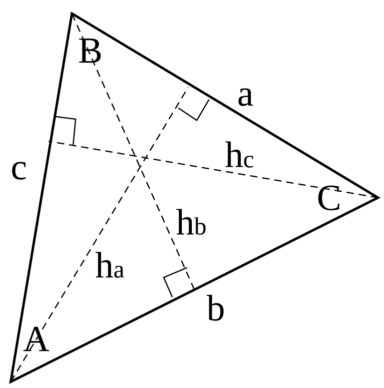


Figure 3: Strip angles, sides and altitudes

Table 4: Angles from dimensions

Angle	**Formula**
A	$arccos(\frac{1+(\frac{h_b}{h_c})^2-(\frac{h_b}{h_a})^2)}{2\frac{h_b}{h_c}})$
B	$arccos(\frac{1+(\frac{h_a}{h_c})^2-(\frac{h_a}{h_b})^2)}{2\frac{h_a}{h_c}})$
C	$arccos(\frac{1+(\frac{h_a}{h_b})^2-(\frac{h_a}{h_c})^2)}{2\frac{h_a}{h_b}})$

5 Planing form settings

Determining the planing form depth for hexagonal rods is of course trivial. For pentagonal and quadrate rods, there are two alternative measures by which planing form settings are described. First is the depth of the groove; in tFigure 4 below for a penta, this is the value P. It is the micrometer measurement from one planed surface to the corner where the enamel meets the opposite planed side. Second is the length of the diagonal marked A in the diagram, which is the same as A in Figure 2. This is the micrometer measurement from the center of the enamel to the opposite corner where the planed surfaces meet. Measurement A is probably easier to execute at the bench. For hexagonal rods, P and A are identical.

Probably very few rod builders hand plane strips for pentagonal or quadrate rods in traditional adjustable forms; a Morgan hand mill or powered mill is more typical. For penta and quad forms use the following figure and table for P or A. The diagram is for one of the mirror image grooves on a pentagonal form. The strip side dimensions H, A and $2O$ refer back to the right triangle diagram in Figure 2.

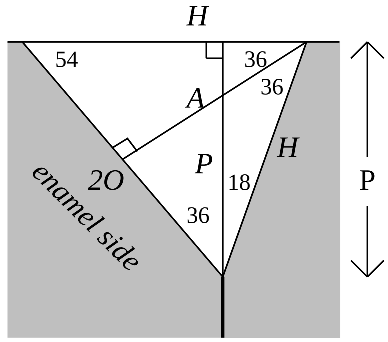


Figure 4: Planing form settings; half of pentagonal form pictured

Table 5: Planing form settings

Geometry	Formula			Compute
Hexagonal	P	$=$	$\frac{D}{2}$	$0.5D$
	A	$=$	$\frac{D}{2}$	$0.5D$
Pentagonal	P	$=$	$\frac{D\sin 72}{1+\cos 36}$	$0.5257D$
	A	$=$	$(1-\frac{1}{1+\cos 36})D$	$0.4472D$
Quadrate	P	$=$	$\frac{D}{2\cos 45}$	$0.7071D$
	A	$=$	$\frac{D}{2}$	$0.5D$

5.1 George Barnes method

Unlike hexagonal forms, indicator points for penta and quad forms are uncommon. An alternative method of setting the forms using a dial indicator and a short length of drill rod was given by George Barns in an article "Setting Tapers in Forms" in *The Planing Form* issue 39 (May/June 1996). In the table below, Q is the diameter of the drill rod and M is the amount the rod sits proud of the form. (It is assumed that the drill rod is large enough to sit proud of the forms.) To use these formulas replace A with its formula in terms of D from Table 3, substituting the known values of D and Q, and solving for M.

Table 6: Barnes formulas for planing form setting

Hexagonal	A	$=$	$1.5Q - M$
Pentagonal	A	$=$	$1.3507Q - 0.8507M$
Quadrate	A	$=$	$0.7071(1.07071Q - M)$

6 Ferrule size

For fitting premade metal ferrules, it is useful to know the diameters of the inside *incircle* and outside *circumcircle*. Calculation of the radius (and diameter) of these circles is straightforward from the triangles shown in Figures 1 and 2. The radius of the incircle is length of the adjacent side (A) and the circumcircle is the length of the hypotenuse (H). Multiply these by two to get ferrule size F and convert to 64-ths to see the options.

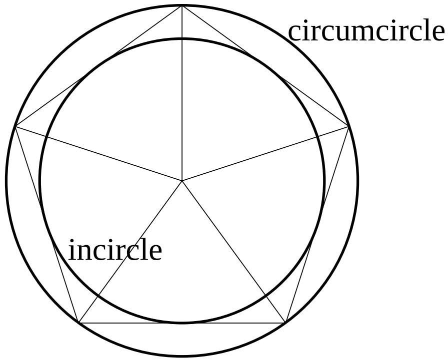


Figure 5: The incircle and circumcircle of a pentagon

Table 7: Ferrule size calculations

Geometry	Circle diameter	Diameter Formula	Diameter Compute
Hexagonal	Incircle	$2A = D$	D
	Circumcircle	$2H = \frac{D}{\cos 30}$	$1.1547D$
Pentagonal	Incircle	$2A = (2 - \frac{2}{1+\cos 36})D$	$0.8944D$
	Circumcircle	$2H = \frac{2D}{1+\cos 36}$	$1.1056D$
Quadrate	Incircle	$2A = D$	D
	Circumcircle	$2H = \frac{D}{\cos 45}$	$1.4142D$

7 Cross sectional area

The cross sectional area at a station can be calculated from formulas for regular polygons, or more easily for us, from the right triangles in Figures 1 and 2. The area of a triangle is $\frac{1}{2} \times base \times height$ where $base$ is the side opposite (O) and $height$ is the side adjacent (A). Values of A and O in terms of dimension D are from Table 3.

Table 8: Cross sectional area

Geometry	Formula	Compute
Hexagonal	$Area = 12 \times \frac{1}{2} \times \frac{D \tan 30}{2} \times \frac{D}{2}$	$0.8660D^2$
Pentagonal	$Area = 10 \times \frac{1}{2} \times \frac{D \sin 36}{1+\cos 36} \times (1 - \frac{1}{1+\cos 36})D$	$0.7265D^2$
Quadrate	$Area = 8 \times \frac{1}{2} \times \frac{D}{2} \times \frac{D}{2}$	D^2

7.1 Converting geometries

One method of converting tapers between different geometries (not necessarily the best) is to equate the cross sectional areas at each station. This comes down to a simple dimension multiplier.

Table 9: Multipliers to convert geometries by equal cross section

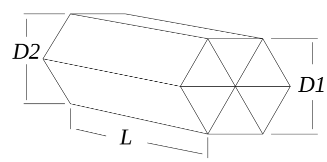
From Geometry	To geometry: Hex	To geometry: Penta	To geometry: Quad
Hexagonal	-	1.0918	0.9306
Pentagonal	0.9159	-	0.8523
Quadrate	1.0746	1.1732	-

7.2 Volume and weight

Garrison gives the formula for a tapered section of hexagonal cane rod; in geometric terms a *fustrum*. It depends on the length of the section L and the areas of the two ends (from Table 8 above). The same general formula applies to all geometries:

$$Volume = \frac{L}{3}(Area_1 + Area_2 + \sqrt{Area_1 \times Area_2})$$

If we need to compute the volume of a short section of a cane rod (say one inch), where the difference in the areas of the two ends is very small, we area able to simplify our calculations because the area term above is approximately equal to $3 \times$ average area. The approximate volume becomes just the average area of the two ends times the length L.



$$Volume \approx \frac{Area_1 + Area_2}{2} \times L$$

Figure 6: Approximate volume of a rod segment

To compute the weight of the section, it is typical to use Garrison's calculated value of 0.668 ounces per cubic inch, but of course other values may be appropriate. The weight of the cane rod, usually for one inch sections, is used in Garrison's stress calculations.

8 Dimension increase due to glue lines

The glue lines will add to the dimension of the finished rod. Each rod geometry leads to a different amount of increase, depending on the number of glue lines and the angles at which they occur. Ray Gould measured a typical glue line and found a nominal thickness of 0.001 inch. (Gould (2005) **Cane Rods: Tips & Tapers**, page 33.) He shows the trig leadings to the result that, for a hexagonal rod, the total dimension inflation is five times the glue line thickness, or typically about 0.005 inches.

Figure 7 shows the angles (θ) at which the glue lines are crossed. The effective thickness of the glue line is $T = \frac{G}{\sin\theta}$ where G is the nominal glue line thickness.

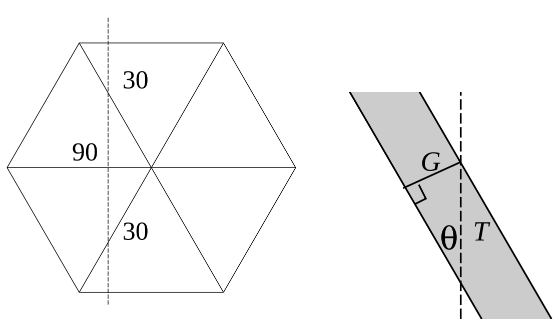


Figure 7: Angles of glue lines

Fortunately, Gould's results are incorrect. Mike McGuire shows that examination of a diagram with (very) exagerated glue lines leads quickly to the correct answer.

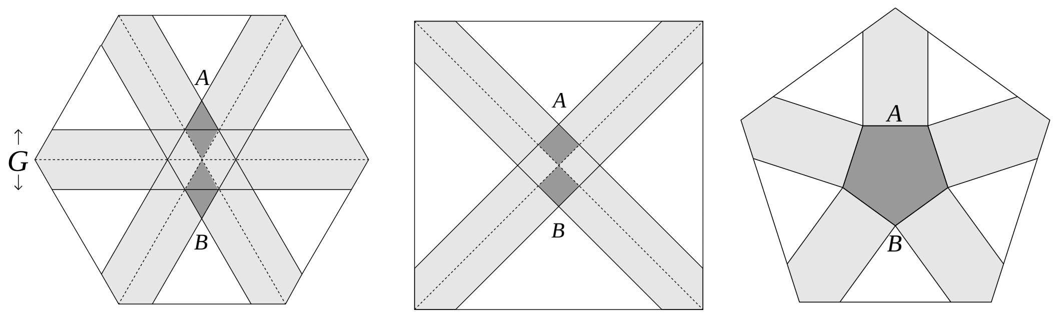


Figure 8: Geometry of glue lines

Starting with the hexaginal cross section, the distance between the strip apexes A and B is the glue line inflation. This is the height of the four stacked shaded triangles in the center. Each triangle has a height of $\frac{1}{2}$ the glue line thickness G. So the total inflation is $2 \times G$. The quad and pentagonal cross sections require a little trig to solve: the quad is $2 \times \frac{1}{\sin 45} \times \frac{1}{2}G$ and the penta is $\frac{1+\cos 36}{2 \sin 36} \times G$ (which is just an application of the formula from Table 3 line (13)).

What was wrong with Gould's analysis? The total thickness of the glue lines crossed in the hexagon of Figure 8 is correct at 5G, but the glue is also separating the strips on the side, pushing the two left strips and the two right in opposite directions. So while there is 5G of glue, there is 3G less cane.

Table 9: Glue line dimension inflation

Geometry	Formula	Compute
Hexagonal	$4 \times \frac{1}{2} \times G$	$2G$
Quadrate	$2 \times \frac{1}{\sin 45} \times \frac{1}{2}G$	$1.4142G$
Pentagonal	$\frac{1+\cos 36}{2 \sin 36} \times G$	$1.5388G$

9 Moments of Inertia

The moment of inertia is the stiffness of a body due to its size and shape. In our case, the shape is the rod geometry and the size is the rod dimension at a point. It is used in calculating the stiffness and deflection of rods. The general formula is $I = a \times D^4$ where a is a constant that depends on the shape.

Roark's Formulas for Stress & Strain, Table 1 section 27 gives a general formula for computing I. Referring back to Figure 2, the the formula involves the length of the sides H and O and the area of the whole cross section A (see Table 8).

$$I = \frac{1}{24}A(6H^2 - 4O^2)$$

Table 11 gives the moment of inerta in terms of rod dimension D:

Table 10. Moments of inertia

Geometry	Compute
Hexagonal	$0.060141D^4$
Pentagonal	$0.042716D^4$
Quadrate	$0.083333D^4$

9.1 Dimension Conversion

These moment of interia formulas can be used to convert dimensions between geometries by equating their stiffness at each point along the rod. For example, to convert a hexagonal to a pentagonal rod, solve the formula for D_{penta} in terms of D_{hex}:

$$\begin{aligned} 0.049839D_{penta}^4 &= 0.060641D_{hex}^4 \\ D_{penta} &= \sqrt[4]{\tfrac{0.060131}{0.042716}}D_{hex} \\ D_{penta} &= 1.089332D_{hex} \end{aligned}$$

This leads to the following multipliers to convert between geometries. If your goal is a rod of new geometry which casts like the original, conversion by equal stiffness is generally advised over equal cross sectional areas from Table 9.

Table 12: Multipliers to convert geometries by equal stiffness

	To geometry		
From Geometry	**Hex**	**Penta**	**Quad**
Hexagonal	-	1.0893	0.9217
Pentagonal	0.9180	-	0.8461
Quadrate	1.0850	1.1818	-

Mike McGuire has a paper **Dimension Compensation for Hollowing Bamboo Rods** on his website that applies the equal stiffness calculations to hollowing.